Chapter 6 : Power Series to solve ODE

Recall : Power series = infinite sum of powers of $(x-a)$
= extension of a polynomial

Almost all functions can be written as power series.

Review of Power Series:

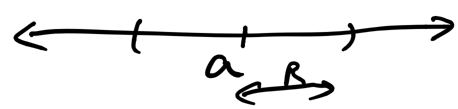
$$\sum_{n=0}^{\infty} c_n (x-a)^n \quad , \quad c_n : \text{coefficient}$$

- Convergence : Partial sum

$$\left(S_N = \sum_{n=0}^{N} c_n (x-a)^n = c_0 + c_1 (x-a) + \cdots + c_N (x-a)^N \right.$$

Series $\to S_0(x)$ iff $\lim\limits_{N\to\infty} S_N = S_0(x)$

- Interval of convergence : set of all x for which the series converges
- Center of the interval of convergence : a



- Radius of Convergence = Radius of the interval $= R$

For $R > 0$, $\begin{cases} |x-a| < R, & \text{series converges abs} \\ |x-a| > R, & \text{— diverges} \end{cases}$

$\S\ |x-a| = R$, check for convergence.

$R = 0$, only converges at $x = a$

$R = \infty$, converges for all x.

- How to determine R?

Convergence Tests: Ratio Test

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L$$

If $L < 1 \Rightarrow$ converges abs $\Rightarrow$ determine R

$L > 1 \Rightarrow$ diverges

$L = 1 \Rightarrow$ inconclusive $\Leftrightarrow$ endpoints

End points: Use other tests: divergence test / alternating series test / p-series test / harmonic series / comparison test / integral tests

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \left| \frac{\left(\frac{(x-3)^{n+1}}{2^{n+1}(n+1)}\right)}{\left(\frac{(x-3)^n}{2^n n}\right)} \right|$

$$= \lim_{n\to\infty} |x-3| \frac{n}{2(n+1)} = |x-3| \lim_{n\to\infty} \frac{n}{2(n+1)}$$

$$= |x-3| \lim_{n\to\infty} \frac{1}{2(1+\frac{1}{n})} = |x-3| \frac{1}{2} = L$$

Find x so that $|x-3|\frac{1}{2} = L < 1$

$\Longleftrightarrow |x-3| < 2$

$\Longleftrightarrow -2 < x-3 < 2$

$\Longleftrightarrow 1 < x < 5$

Radius of convergence = 2

Series $\begin{cases} \text{converges abs for } 1 < x < 5 \\ \text{diverges for } x < 1,\ x > 5 \end{cases}$

End points: ⊕ $x = 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{(1-3)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

converges by the alternating series test

⊕ $x = 5$, $\displaystyle\sum_{n=1}^{\infty} \frac{(5-3)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$

(p series $(p = 1)$ = harmonic series)
diverges.

Interval of convergence = $[1, 5)$

Summary: Ratio Test → Radius
Another test to check end points

⊕ Functions → Power Series
Taylor Series / Mac laurin series
$(x-a)$ $(x-0)$

If f is analytic (differentiable at any order)
then at $x=0$

$$f(x) \cong \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

(equality holds only when the series converges)

$$c_n = \frac{f^{(n)}(0)}{n!}$$

Ex: $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$

$f^{(n)}(x) = e^x$

$$c_n = \frac{e^0}{n!} = \frac{1}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$ converges for all x

$$\Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$\cos x = \dots$

$\sin x = \dots$

$\ln(1+x) = \dots$

$\frac{1}{1-x} = \dots$

(+) Solving ODE by Power Series

Ex: $\underbrace{y' + y = 0}$ (ODE)

Step 1: Write y as a power series

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Step 2: Compute y' (y'') in terms of a power series

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

Step 3: Do algebra with power series

$$y + y' = \sum_{n=0}^{\infty} c_n x^n + \sum_{n=1}^{\infty} c_n n x^{n-1}$$

Power difference $\Rightarrow$ modification needed
= shifting of indices

Goal: These power series involve the same power of x!

$m = n-1$, $n = m+1$ } aux step

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = \sum_{m=0}^{\infty} c_{m+1}(m+1)x^m$$

$$= \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n$$

$$0 = y' + y = \sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n$$

$$= \sum_{n=0}^{\infty} [\underbrace{c_n + c_{n+1}(n+1)}_{\text{new coefficient}}] x^n$$

$\Leftrightarrow$ $c_n + c_{n+1}(n+1) = 0$ for every n
(Recursive relation)

⊕ <u>Step 4</u> : Solving that recursive relation
(finding a pattern)

$c_0 = C, \quad c_0 + c_1(1) = 0 \Rightarrow c_1 = -c_0$

$c_1 = -C, \quad c_1 + c_2(2) = 0 \Rightarrow c_2 = -\frac{1}{2}c_1$

$c_2 = \frac{C}{2}, \quad c_2 + c_3(3) = 0 \Rightarrow c_3 = -\frac{1}{3}c_2$

$c_3 = -\frac{C}{3 \cdot 2}, \quad c_3 + c_4(4) = 0 \Rightarrow c_4 = -\frac{1}{4}c_3$

$$c_4 = \frac{C}{4 \cdot 3 \cdot 2} = \frac{C}{4!}$$

$$c_n = \frac{C(-1)^n}{n!}$$

Solution : $y = \sum_{n=0}^{\infty} c_n x^n = \underbrace{C\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n}_{\text{represent a function}}$

Ce^{-x}

Check : $y = Ce^{-x}, \quad y' = -Ce^{-x}$

$y + y' = 0!$